

Propagators associated to periodic Hamiltonians: an example of the Aharonov-Bohm Hamiltonian with two vortices

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Abstract

We consider an invariant quantum Hamiltonian $H = -\Delta_{LB} + V$ in the L^2 space based on a Riemannian manifold \tilde{M} with a discrete symmetry group Γ . Typically, \tilde{M} is the universal covering space of a multiply connected manifold M and Γ is the fundamental group of M . To any unitary representation Λ of Γ one can relate another operator on $M = \tilde{M}/\Gamma$, called H_Λ , which formally corresponds to the same differential operator as H but which is determined by quasi-periodic boundary conditions. We give a brief review of the Bloch decomposition of H and of a formula relating the propagators associated to the Hamiltonians H_Λ and H . Then we concentrate on the example of the Aharonov-Bohm effect with two vortices. We explain in detail the construction of the propagator in this case and indicate all essential intermediate steps.

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1 Introduction

Suppose that there is given a connected Riemannian manifold \tilde{M} with a discrete symmetry group Γ . Let us consider a Γ -periodic Hamilton operator in $L^2(\tilde{M})$ of the form $H = -\Delta_{LB} + V$ where Δ_{LB} is the Laplace-Beltrami operator and V is a Γ -invariant bounded real function on \tilde{M} . To any unitary representation Λ of Γ one can relate another operator on $M = \tilde{M}/\Gamma$, called H_Λ , which formally corresponds to the same differential operator as H but which is determined by quasi-periodic boundary conditions. In the framework of the Feynman path integral there was derived a remarkable formula relating the propagators $\mathcal{K}_t^\Lambda(x, x_0)$ and $\mathcal{K}_t(x, x_0)$ associated respectively to the Hamiltonians H_Λ and H [1, 2]. An analogous formula is also known for heat kernels [3]. There exists also an opposite point of view when one decomposes the operator H into a direct integral with components H_Λ where Λ runs over all irreducible unitary

representations of Γ [4, 5, 6]. The evolution operator then decomposes correspondingly. This type of decomposition is an essential step in the Bloch analysis. Let us also note that an alternative approach to the Bloch analysis, based on a more algebraic point of view, has been proposed recently in [7].

The both relations, the propagator formula on the one hand and the generalized Bloch decomposition on the other hand, are in a sense mutually inverse [8]. In the current paper we give a brief review of basic results concerning this relationship. The main purpose of this contribution is, however, a detailed discussion of an application of the formula for propagators. We consider the example of the Aharonov-Bohm effect with two vortices. In this case \tilde{M} is identified with the universal covering space of the plane with two excluded points and Γ is the fundamental group of the same manifold. This problem has been already treated by one of the authors quite a long time ago in [9]. But even in more recent papers one can encounter a discussion of the problem itself [10] as well as of the involved methods [11, 12]. It should be stressed that article [9] is a very brief letter in which only the final formula is presented without any detailed hints about its derivation. This is so even though some steps in the derivation are rather complicated and in no way obvious. Since these details were never published anywhere and since without them the resulting formula may look a bit obscure we aim here to fill in this gap by explaining the approach more carefully and by indicating the necessary intermediate steps.

The paper is organized as follows. In Section 2 we give a brief review of basic results concerning the relationship between the generalized Bloch analysis and the formula for propagators associated to periodic Hamiltonians. In Section 3 we explain the construction of the propagator on the universal covering space in the case of the Aharonov-Bohm effect with two vortices. In Section 4 we discuss the application of the propagator formula in this particular case.

2 Propagators associated to periodic Hamiltonians

2.1 Periodic Hamiltonians

Let \tilde{M} be a connected Riemannian manifold with a discrete and at most countable symmetry group Γ . The action of Γ on \tilde{M} is assumed to be smooth, free and proper (also called properly discontinuous). Denote by $\tilde{\mu}$ the measure on \tilde{M} induced by the Riemannian metric. The quotient $M = \tilde{M}/\Gamma$ is a connected Riemannian manifold with an induced measure μ . This way one gets a principal fiber bundle $\pi : \tilde{M} \rightarrow M$ with the structure group Γ . The L^2 spaces on the manifolds M and \tilde{M} are everywhere tacitly understood with the measures μ and $\tilde{\mu}$, respectively.

Typically, \tilde{M} is the universal covering space of M and $\Gamma = \pi_1(M)$ is the fundamental group of M . For example, this is the case when one is considering the Aharonov-Bohm effect.

To a unitary representation Λ of Γ in a separable Hilbert space \mathcal{L}_Λ one relates the Hilbert space \mathcal{H}_Λ formed by Λ -equivariant vector-valued functions on \tilde{M} . This means

that any function $\psi \in \mathcal{H}_\Lambda$ is measurable with values in \mathcal{L}_Λ and satisfies

$$\forall s \in \Gamma, \psi(s \cdot y) = \Lambda(s)\psi(y) \text{ almost everywhere on } \tilde{M}.$$

Moreover, the norm of ψ induced by the scalar product is finite. If $\psi_1, \psi_2 \in \mathcal{H}_\Lambda$ then the function $y \mapsto \langle \psi_1(y), \psi_2(y) \rangle$ defined on \tilde{M} is Γ -invariant and so it projects to a function s_{ψ_1, ψ_2} defined on M , and the scalar product is defined by

$$\langle \psi_1, \psi_2 \rangle = \int_M s_{\psi_1, \psi_2}(x) d\mu(x).$$

As already announced, our discussion concerns Γ -periodic Hamiltonians on \tilde{M} of the form $H = -\Delta_{LB} + V$ where Δ_{LB} is the Laplace-Beltrami operator and $V(y)$ is a Γ -invariant measurable bounded real function on \tilde{M} . Here we accept the Friedrichs extension as the preferred self-adjoint extension of semibounded symmetric operators defined on test functions.

To the same differential operator, $-\Delta_{LB} + V$, one can relate a selfadjoint operator H_Λ in the space \mathcal{H}_Λ for any unitary representation Λ of Γ . Let us define $\Phi_\Lambda : C_0^\infty(\tilde{M}) \otimes \mathcal{L}_\Lambda \rightarrow \mathcal{H}_\Lambda$ by

$$\forall \varphi \in C_0^\infty(\tilde{M}), \forall v \in \mathcal{L}_\Lambda, (\Phi_\Lambda \varphi \otimes v)(y) = \sum_{s \in \Gamma} \varphi(s \cdot y) \Lambda(s^{-1})v.$$

Since the action of Γ is proper, the vector-valued function $\Phi_\Lambda \varphi \otimes v$ is smooth. Moreover, $\Phi_\Lambda \varphi \otimes v$ is Λ -equivariant and the norm of $\Phi_\Lambda \varphi \otimes v$ in \mathcal{H}_Λ is finite. Furthermore, the range of Φ_Λ is dense in \mathcal{H}_Λ . The Laplace-Beltrami operator is well defined on $\text{Ran}(\Phi_\Lambda)$ and it holds

$$\Delta_{LB} \Phi_\Lambda[\varphi \otimes v] = \Phi_\Lambda[\Delta_{LB} \varphi \otimes v].$$

One can also verify that the differential operator $-\Delta_{LB}$ is positive on the domain $\text{Ran}(\Phi_\Lambda) \subset \mathcal{H}_\Lambda$. Since the function $V(y)$ is Γ -invariant, the multiplication operator by V is well defined in the Hilbert space \mathcal{H}_Λ . The Hamiltonian H_Λ is defined as the Friedrichs extension of the differential operator $-\Delta_{LB} + V$ considered on the domain $\text{Ran} \Phi_\Lambda$.

2.2 A generalization of the Bloch analysis

Let $\hat{\Gamma}$ be the dual space to Γ (the quotient space of the space of irreducible unitary representations of Γ). In the first step of the generalized Bloch analysis one decomposes H into a direct integral over $\hat{\Gamma}$ with the components being equal to H_Λ . As a corollary one obtains a similar relationship for the evolution operators $U(t) = \exp(-itH)$ and $U_\Lambda(t) = \exp(-itH_\Lambda)$, $t \in \mathbb{R}$. To achieve this goal a well defined harmonic analysis on the group Γ is necessary.

It is known that the harmonic analysis is well established for locally compact groups of type I [13]. So all formulas presented bellow are perfectly well defined provided Γ is a type I group. A countable discrete group is type I, however, if and only if it has

an Abelian normal subgroup of finite index [14, Satz 6]. This means that there exist multiply connected configuration spaces of interest whose fundamental groups are not of type I. For example, the fundamental group in the case of the Aharonov-Bohm effect with two vortices is the free group with two generators and it is not of type I. Fortunately, in this case, too, there exists a well defined harmonic analysis [15].

Let us recall the basic properties of the harmonic analysis on discrete type I groups [13]. In that case the Haar measure on Γ is chosen as the counting measure. Let $d\hat{m}$ be the Plancherel measure on $\hat{\Gamma}$. Denote by $\mathcal{I}_2(\mathcal{L}_\Lambda) \equiv \mathcal{L}_\Lambda^* \otimes \mathcal{L}_\Lambda$ the Hilbert space formed by Hilbert-Schmidt operators on \mathcal{L}_Λ (\mathcal{L}_Λ^* is the dual space to \mathcal{L}_Λ). The Fourier transformation is defined as a unitary mapping

$$\mathcal{F} : L^2(\Gamma) \rightarrow \int_{\hat{\Gamma}}^{\oplus} \mathcal{I}_2(\mathcal{L}_\Lambda) d\hat{m}(\Lambda).$$

For $f \in L^1(\Gamma) \subset L^2(\Gamma)$ one has

$$\mathcal{F}[f](\Lambda) = \sum_{s \in \Gamma} f(s) \Lambda(s).$$

Conversely, if f is of the form $f = g * h$ (the convolution) where $g, h \in L^1(\Gamma)$, and $\hat{f} = \mathcal{F}[f]$ then

$$f(s) = \int_{\hat{\Gamma}} \text{Tr}[\Lambda(s)^* \hat{f}(\Lambda)] d\hat{m}(\Lambda).$$

It is known that if Γ is a countable discrete group of type I then $\dim \mathcal{L}_\Lambda$ is a bounded function of Λ on the dual space $\hat{\Gamma}$ [14, Korollar I]. Using the unitarity of the Fourier transform one finds that

$$\hat{m}(\hat{\Gamma}) \leq \int_{\hat{\Gamma}} \dim \mathcal{L}_\Lambda d\hat{m}(\Lambda) = 1.$$

The following rule satisfied by the Fourier transformation is also of crucial importance:

$$\forall s \in \Gamma, \forall f \in L^2(\Gamma), \mathcal{F}[f(s \cdot g)](\Lambda) = \Lambda(s^{-1}) \mathcal{F}[f(g)](\Lambda).$$

Now we are going to construct a unitary mapping

$$\Phi : L^2(\tilde{M}) \rightarrow \int_{\hat{\Gamma}}^{\oplus} \mathcal{L}_\Lambda^* \otimes \mathcal{H}_\Lambda d\hat{m}(\Lambda)$$

which makes it possible to decompose the Hamiltonian H . Observe that the tensor product $\mathcal{L}_\Lambda^* \otimes \mathcal{H}_\Lambda$ can be naturally identified with the Hilbert space of $1 \otimes \Lambda$ -equivariant operator-valued functions on \tilde{M} with values in $\mathcal{L}_\Lambda^* \otimes \mathcal{L}_\Lambda \equiv \mathcal{I}_2(\mathcal{L}_\Lambda)$. For $f \in L^2(\tilde{M})$ and $y \in \tilde{M}$ set

$$\forall s \in \Gamma, f_y(s) = f(s^{-1} \cdot y).$$

The norm $\|f_y\|$ in $L^2(\Gamma)$ is a Γ -invariant function of $y \in \tilde{M}$, and the projection of this function onto M can be checked to be square integrable. Hence for almost all $x \in M$

and all $y \in \pi^{-1}(\{x\})$ it holds $f_y \in L^2(\Gamma)$. We define the component $\Phi[f](\Lambda)$, $\Lambda \in \hat{\Gamma}$, by the prescription

$$\Phi[f](\Lambda)(y) := \mathcal{F}[f_y](\Lambda) \in \mathcal{I}_2(\mathcal{L}_\Lambda).$$

In particular, if $f \in L^1(\tilde{M}) \cap L^2(\tilde{M})$ then

$$\Phi[f](\Lambda)(y) = \sum_{s \in \Gamma} f(s^{-1} \cdot y) \Lambda(s).$$

Equivalently one can define Φ in the following way. For $\varphi \in C_0^\infty(\tilde{M})$, $v \in \mathcal{L}_\Lambda$ and $y \in \tilde{M}$ set

$$\Phi[\varphi](\Lambda)(y)v = (\Phi_\Lambda \varphi \otimes v)(y). \quad (1)$$

Then Φ introduced in (1) is an isometry and extends unambiguously to a unitary mapping.

Finally one can verify the formula

$$\Phi H \Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} 1 \otimes H_\Lambda d\hat{m}(\Lambda)$$

which represents the sought Bloch decomposition. As a corollary we have

$$\Phi U(t) \Phi^{-1} = \int_{\hat{\Gamma}}^{\oplus} 1 \otimes U_\Lambda(t) d\hat{m}(\Lambda). \quad (2)$$

2.3 A construction for propagators associated to periodic Hamiltonians

In equality (2), the evolution operator $U(t)$ is expressed in terms of $U_\Lambda(t)$, $\Lambda \in \hat{\Gamma}$. It is possible to invert this relationship and to derive a formula for the propagator associated to H_Λ which is expressed in terms of the propagator associated to H .

The propagators are regarded as distributions which are introduced as kernels of the corresponding evolution operators. Recall that by the Schwartz kernel theorem (see, for example, [16, Theorem 5.2.1]), to every $B \in \mathcal{B}(L^2(\tilde{M}))$ there exists one and only one $\beta \in \mathcal{D}'(\tilde{M} \times \tilde{M})$ such that

$$\forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{M}), \quad \beta(\overline{\varphi_1} \otimes \varphi_2) = \langle \varphi_1, B \varphi_2 \rangle.$$

Moreover, the map $B \mapsto \beta$ is injective. One calls β the kernel of B .

The kernel theorem can be extended to Hilbert spaces formed by Λ -equivariant vector-valued functions. In this case the kernels are operator-valued distributions. To every $B \in \mathcal{B}(\mathcal{H}_\Lambda)$ there exists one and only one $\beta \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(\mathcal{L}_\Lambda)$ such that

$$\begin{aligned} \forall \varphi_1, \varphi_2 \in C_0^\infty(\tilde{M}), \forall v_1, v_2 \in \mathcal{L}_\Lambda, \\ \langle v_1, \beta(\overline{\varphi_1} \otimes \varphi_2) v_2 \rangle = \langle \Phi_\Lambda \varphi_1 \otimes v_1, B \Phi_\Lambda \varphi_2 \otimes v_2 \rangle. \end{aligned}$$

The distribution β is Λ -equivariant:

$$\forall s \in \Gamma, \quad \beta(s \cdot y_1, y_2) = \Lambda(s) \beta(y_1, y_2), \quad \beta(y_1, s \cdot y_2) = \beta(y_1, y_2) \Lambda(s^{-1})$$

In this case, too, the map $B \mapsto \beta$ is injective.

Denote by $\mathcal{K}_t \in \mathcal{D}'(\tilde{M} \times \tilde{M})$ the kernel of $U(t) \in \mathcal{B}(L^2(\tilde{M}))$, and by $\mathcal{K}_t^\Lambda \in \mathcal{D}'(\tilde{M} \times \tilde{M}) \otimes \mathcal{B}(\mathcal{L}_\Lambda)$ the kernel of $U_\Lambda(t) \in \mathcal{B}(\mathcal{H}_\Lambda)$. Here and everywhere in this section, t is a real parameter. The kernel \mathcal{K}_t^Λ is Λ -equivariant:

$$\forall s \in \Gamma, \quad \mathcal{K}_t^\Lambda(s \cdot y_1, y_2) = \Lambda(s) \mathcal{K}_t^\Lambda(y_1, y_2), \quad \mathcal{K}_t^\Lambda(y_1, s \cdot y_2) = \mathcal{K}_t^\Lambda(y_1, y_2) \Lambda(s^{-1}).$$

First we rewrite the Bloch decomposition of the propagator (2) in terms of kernels. It is possible to prove that, for all $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{M})$, the function $\Lambda \mapsto \text{Tr}[\mathcal{K}_t^\Lambda(\overline{\varphi_1} \otimes \varphi_2)]$ is integrable on $\hat{\Gamma}$ and

$$\mathcal{K}_t(\varphi_1 \otimes \varphi_2) = \int_{\hat{\Gamma}} \text{Tr}[\mathcal{K}_t^\Lambda(\varphi_1 \otimes \varphi_2)] d\hat{m}(\Lambda).$$

An inverse relation was derived by Schulman in the framework of path integration [1, 2] and reads

$$\mathcal{K}_t^\Lambda(x, y) = \sum_{s \in \Gamma} \Lambda(s) \mathcal{K}_t(s^{-1} \cdot x, y). \quad (3)$$

It is possible to give (3) the following rigorous interpretation. Suppose that $\varphi_1, \varphi_2 \in C_0^\infty(\tilde{M})$ are fixed but otherwise arbitrary. Set

$$F_t(s) = \mathcal{K}_t(\varphi_1(s^{-1} \cdot y_1) \otimes \varphi_2(y_2)) \text{ for } s \in \Gamma,$$

and

$$G_t(\Lambda) = \mathcal{K}_t^\Lambda(\varphi_1 \otimes \varphi_2) \in \mathcal{J}_2(\mathcal{L}_\Lambda) \text{ for } \Lambda \in \hat{\Gamma}.$$

One can show that $F_t \in L^2(\Gamma)$ and G_t is bounded on $\hat{\Gamma}$ in the Hilbert-Schmidt norm. Recalling that $\hat{m}(\hat{\Gamma}) \leq 1$ we have $\|G_t(\cdot)\| \in L^1(\hat{\Gamma}) \cap L^2(\hat{\Gamma})$. In [8] it is verified that

$$F_t = \mathcal{F}^{-1}[G_t].$$

and, consequently,

$$G_t = \mathcal{F}[F_t]. \quad (4)$$

Rewriting (4) formally gives equality (3).

3 The Aharonov-Bohm effect with two vortices: the propagator on the universal covering space

3.1 A formula for the propagator

The configuration space for the Aharonov-Bohm effect with two vortices is the plane with two excluded points, $M = \mathbb{R}^2 \setminus \{a, b\}$. This is a flat Riemannian manifold and the same is true for the universal covering space \tilde{M} . Let $\pi : \tilde{M} \rightarrow M$ be the projection. It is convenient to complete the manifold \tilde{M} by a countable set of points $\mathcal{A} \cup \mathcal{B}$ which lie on the border of \tilde{M} and project onto the excluded points, $\pi(\mathcal{A}) = \{a\}$ and $\pi(\mathcal{B}) = \{b\}$.

\tilde{M} looks locally like \mathbb{R}^2 but differs from the Euclidean space by some global features. First of all, not every two points from \tilde{M} can be connected by a geodesic segment. Fix a point $x \in \tilde{M}$. The symbol $D(x)$, as introduced below in (15), stands for the set of points $y \in \tilde{M}$ which can be connected with x by a geodesic segment. The domain $D(x)$ is one sheet of the covering $\tilde{M} \rightarrow M$. It can be identified with \mathbb{R}^2 cut along two halflines with the limit points a and b , respectively. Thus the border $\partial D(x)$ is formed by four halflines. The universal covering space \tilde{M} can be imagined as a result of an infinite process of glueing together countably many copies of $D(x)$ with each copy having four neighbors.

The fundamental group of M , called Γ , is known to be the free group with two generators g_a and g_b . For the generator g_a one can choose the homotopy class of a simple positively oriented loop winding once around the point a and leaving the point b in the exterior. Analogously one can choose the generator g_b by interchanging the role of a and b . One-dimensional unitary representations Λ of Γ are determined by two numbers α, β , $0 \leq \alpha, \beta < 1$, such that

$$\Lambda(g_a) = e^{2\pi i \alpha}, \quad \Lambda(g_b) = e^{2\pi i \beta}.$$

The standard way to define the Aharonov-Bohm Hamiltonian with two vortices is to choose a vector potential \vec{A} for which $\text{rot } \vec{A} = 0$ on M and such that the nonintegrable phase factor [17] for a closed path from the homotopy class g_a or g_b equals $e^{2\pi i \alpha}$ or $e^{2\pi i \beta}$, respectively (assuming that $0 < \alpha, \beta < 1$). The Hamiltonian then acts as the differential operator $(-i\nabla - \vec{A})^2$ in $L^2(M)$. A unitarily equivalent and for our purposes more convenient possibility is to work with the Hamiltonian $H_\Lambda = -\Delta$ in the Hilbert space \mathcal{H}_Λ of Λ -equivariant functions on \tilde{M} , as introduced in Section 2.1. Parallely one considers the free Hamiltonian $H = -\Delta$ in $L^2(\tilde{M})$. H is Γ -periodic. In order to compute, according to prescription (3), the propagator $\mathcal{K}^\Lambda(t, x, y)$ associated to H_Λ one needs to derive a formula for the free propagator $\mathcal{K}(t, x, y)$ on \tilde{M} . Such a formula is recalled below following [9].

Let ϑ be the Heaviside step function. For $x, y \in \tilde{M} \cup \mathcal{A} \cup \mathcal{B}$ set $\chi(x, y) = 1$ if the points x, y can be connected by a geodesic segment, and $\chi(x, y) = 0$ otherwise. Given in addition $t \in \mathbb{R}$ we define

$$Z(t, x, y) = \vartheta(t) \chi(x, y) \frac{1}{4\pi i t} \exp\left(\frac{i}{4t} \text{dist}^2(x, y)\right),$$

Furthermore, for $x_1, x_2, x_3 \in \tilde{M} \cup \mathcal{A} \cup \mathcal{B}$ such that $\chi(x_1, x_2) = \chi(x_2, x_3) = 1$, and for $t_1, t_2 > 0$ we set

$$V\left(\begin{array}{c} x_3, x_2, x_1 \\ t_2, t_1 \end{array}\right) = 2i \left(\left(\theta - \pi + i \log\left(\frac{t_2 r_1}{t_1 r_2}\right) \right)^{-1} - \left(\theta + \pi + i \log\left(\frac{t_2 r_1}{t_1 r_2}\right) \right)^{-1} \right)$$

where $\theta = \angle x_1, x_2, x_3 \in \mathbb{R}$ is the oriented angle and $r_1 = \text{dist}(x_1, x_2)$, $r_2 = \text{dist}(x_2, x_3)$. Note that if the inner vertex x_2 belongs to the set of extreme points $\mathcal{A} \cup \mathcal{B}$ then the angle θ can take any real value.

We claim that the free propagator on \tilde{M} equals

$$\mathcal{K}(t, x, x_0) = \sum_{\gamma} \mathcal{K}_{\gamma}(t, x, x_0) \quad (5)$$

where the sum runs over all piecewise geodesic curves $\gamma : x_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n \rightarrow x$ with the inner vertices C_j , $1 \leq j \leq n$, belonging to the set of extreme points $\mathcal{A} \cup \mathcal{B}$. This means that it should hold $\chi(x_0, C_1) = \chi(C_1, C_2) = \dots = \chi(C_n, x) = 1$. Let us denote by $|\gamma| = n$ the length of the sequence (C_1, C_2, \dots, C_n) . In particular, if $|\gamma| = 0$ then γ designates the geodesic segment $x_0 \rightarrow x$. To simplify notation we set everywhere where convenient $C_0 = x_0$ and $C_{n+1} = x$. With this convention, the summands in (5) equal

$$\begin{aligned} & \mathcal{K}_{\gamma}(t, x, x_0) \\ &= \int_{\mathbb{R}^{n+1}} dt_n \dots dt_0 \delta(t_n + \dots + t_0 - t) \prod_{j=0}^{n-1} V \left(\begin{array}{c} C_{j+2}, C_{j+1}, C_j \\ t_{j+1}, t_j \end{array} \right) \prod_{j=0}^n Z(t_j, C_{j+1}, C_j). \end{aligned} \quad (6)$$

In particular, if $|\gamma| = 0$ then $\mathcal{K}_{\gamma}(t, x, x_0) = Z(t, x, x_0)$, and if $|\gamma| = 1$ then γ designates a path composed of two geodesic segments $x_0 \rightarrow C \rightarrow x$, with $C \in \mathcal{A} \cup \mathcal{B}$, and

$$\mathcal{K}_{\gamma}(t, x, x_0) = \vartheta(t) \int_0^t V \left(\begin{array}{c} x, C, x_0 \\ t-s, s \end{array} \right) Z(t-s, x, C) Z(s, C, x_0) ds.$$

3.2 Auxiliary relations

As is well known, in \mathbb{R}^2 it holds true that

$$\left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \frac{1}{x + iy} = 2\pi \delta(x) \delta(y)$$

and, consequently,

$$\Delta \frac{1}{x + iy} = 2\pi (\delta(y) \delta'(x) - i \delta(x) \delta'(y)).$$

With the aid of the last equality one can verify that the relation

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\theta + i \log \left(\frac{t}{r} \right) \right)^{-1} = \frac{2\pi t}{r^2} (\delta(t-r) \delta'(\theta) - i r \delta'(t-r) \delta(\theta)) \quad (7)$$

is valid on the domain $t > 0$, $r > 0$, $\theta \in \mathbb{R}$. It is straightforward to see that, on the same domain,

$$\left(r \frac{\partial}{\partial r} + t \frac{\partial}{\partial t} \right) \left(\theta + i \log \left(\frac{t}{r} \right) \right)^{-1} = 0. \quad (8)$$

Combining (7), (8) and the Leibniz rule one finds that

$$\begin{aligned} & \left(i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \left(\theta + i \log \left(\frac{t}{r} \right) \right)^{-1} \frac{1}{t} \exp \left(i \frac{r^2}{4t} \right) \\ &= \frac{2\pi}{r^2} \exp \left(i \frac{r^2}{4t} \right) (\delta(t-r) \delta'(\theta) - i r \delta'(t-r) \delta(\theta)). \end{aligned} \quad (9)$$

Equipped with (9) one can prove the equality

$$\begin{aligned} & \left(i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \int_0^t \left(\theta + i \log \left(\frac{(t-s)r_0}{sr} \right) \right)^{-1} \\ & \quad \times \frac{1}{t-s} \exp \left(i \frac{r^2}{4(t-s)} \right) f(s) ds \\ &= \frac{2\pi r_0}{r^2(r+r_0)} \exp \left(i \frac{(r+r_0)r}{4t} \right) \left[f \left(\frac{tr_0}{r+r_0} \right) \delta'(\theta) \right. \\ & \quad \left. - i \frac{r}{r+r_0} \left(\left(1 + i \frac{r_0(r+r_0)}{4t} \right) f \left(\frac{tr_0}{r+r_0} \right) + \frac{tr_0}{r+r_0} f' \left(\frac{tr_0}{r+r_0} \right) \right) \delta(\theta) \right] \end{aligned} \quad (10)$$

which is true in the sense of distributions for any $r_0 > 0$ and $f \in C^1([0, +\infty[)$, again on the domain $t > 0, r > 0, \theta \in \mathbb{R}$. Note that

$$\frac{1}{\varepsilon} \exp \left(i \frac{r^2}{4\varepsilon} \right) \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0+ \quad \text{in } \mathcal{D}'([0, +\infty[).$$

In particular, letting

$$f(s) = \frac{1}{s} \exp \left(i \frac{r_0^2}{4s} \right)$$

one derives the following equality which is true in the sense of distributions,

$$\begin{aligned} & \left(i \frac{\partial}{\partial t} + \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \int_0^t \left(\theta + i \log \left(\frac{(t-s)r_0}{sr} \right) \right)^{-1} \\ & \quad \times \frac{1}{(t-s)s} \exp \left(i \left(\frac{r^2}{4(t-s)} + \frac{r_0^2}{4s} \right) \right) ds \\ &= \frac{2\pi}{tr^2} \exp \left(i \frac{(r+r_0)^2}{4t} \right) \delta'(\theta). \end{aligned} \quad (11)$$

Let us also recall the following basic fact concerning the generalized Laplacian. If $G \subset \tilde{M}$ is an open set with a piecewise smooth boundary, χ_G is the characteristic function of G , \vec{n} is the normalized outer normal vector field on ∂G and η is a smooth function on \tilde{M} then, in the sense of distributions,

$$\Delta(\eta \chi_G) = (\Delta \eta) \chi_G - \frac{\partial \eta}{\partial \vec{n}} \delta_{\partial G} - \frac{\partial}{\partial \vec{n}} (\eta \delta_{\partial G}). \quad (12)$$

The distribution $\delta_{\partial G}$ is the layer supported on the curve ∂G which is defined by the curve integral

$$\forall \varphi \in C_0^\infty(\tilde{M}), \quad \delta_{\partial G}(\varphi) = \int_{\partial G} \varphi d\ell.$$

3.3 Verification of the formula

We have to show that, for $x_0 \in \tilde{M}$ fixed, the propagator $\mathcal{K}(t, x, x_0)$ defined in (5), (6) verifies the condition

$$\left(i\frac{\partial}{\partial t} + \Delta\right) \mathcal{K}(t, x, x_0) = i\delta(t)\delta(x, x_0) \quad \text{on } \mathbb{R} \times \tilde{M}.$$

This is equivalent to showing that

$$\lim_{t \rightarrow 0_+} \mathcal{K}(t, x, x_0) = \delta(x, x_0) \quad (13)$$

and

$$\left(i\frac{\partial}{\partial t} + \Delta\right) \mathcal{K}(t, x, x_0) = 0 \quad \text{for } t > 0, x \in \tilde{M}. \quad (14)$$

Equality (13) is rather obvious. Since $Z(t, x, x_0)$ looks on the sheet $\{x; \chi(x, x_0) = 1\}$ as the free propagator on \mathbb{R}^2 we have

$$\lim_{t \rightarrow 0_+} Z(t, x, x_0) = \delta(x, x_0).$$

From a similar reason, $\lim_{t \rightarrow 0_+} Z(t, x, C) = 0$ if $C \in \mathcal{A} \cup \mathcal{B}$ and x runs over \tilde{M} . Hence

$$\lim_{t \rightarrow 0_+} \mathcal{K}_\gamma(t, x, x_0) = 0 \quad \text{if } |\gamma| \geq 1.$$

Let us proceed to the verification of (14). First we introduce some notation related to the geometry of the universal covering space \tilde{M} . Denote by ϱ the distance $\text{dist}(a, b)$. Observe that if $C_1, C_2 \in \mathcal{A} \cup \mathcal{B}$ then $\chi(C_1, C_2) = 1$ if and only if $\text{dist}(C_1, C_2) = \varrho$. If this is the case then necessarily $C_1 \in \mathcal{A}$ and $C_2 \in \mathcal{B}$ or vice versa.

For $x \in \tilde{M} \cup \mathcal{A} \cup \mathcal{B}$ set

$$D(x) = \left\{y \in \tilde{M}; \chi(x, y) = 1\right\}. \quad (15)$$

If $x \in \tilde{M}$ then $D(x)$ can be identified with the aid of the projection $\pi : \tilde{M} \rightarrow M$ with the plane cut along two halflines with the limit points a and b , respectively. Thus the border of $D(x)$ consists of two pairs of halflines. One pair has a common limit point $A \in \mathcal{A}$ and is denoted $\partial D(x; A)$, the other pair has a common limit point $B \in \mathcal{B}$ and is denoted $\partial D(x; B)$. We have

$$\partial D(x) = \partial D(x; A) \cup \partial D(x; B). \quad (16)$$

If $C \in \mathcal{A} \cup \mathcal{B}$ then $D(C)$ resembles the universal covering space in the one-vortex case. It can be viewed as a union of countably many sheets glued together in a staircase-like way. Each sheet contributes to the border of $D(C)$ by a pair of halflines with a common limit point C' . Thus the border $\partial D(C)$ is formed by a countable union of pairs of halflines: if $C \in \mathcal{A}$ then we write

$$\partial D(C) = \bigcup_{C' \in \mathcal{B}, \text{dist}(C, C') = \varrho} \partial D(C; C'), \quad (17)$$

if $C \in \mathcal{B}$ then

$$\partial D(C) = \bigcup_{C' \in \mathcal{A}, \text{dist}(C, C') = \varrho} \partial D(C; C'). \quad (18)$$

Let us first examine the case $|\gamma| = 0$. It holds

$$\left(i \frac{\partial}{\partial t} + \Delta\right) Z(t, x, x_0) = 0.$$

for $t > 0$ and $x \in D(x_0)$. Observe also that

$$\frac{\partial}{\partial \vec{n}} Z(t, x, x_0) = 0 \quad \text{for } x \in \partial D(x_0)$$

where \vec{n} is the normalized outer normal vector field on the border $\partial D(x_0)$. This is so since, in the polar coordinates centered at x_0 , $Z(t, x, x_0)$ does not depend on the angle variable. Let us also note that the function $Z(t, x, x_0)$ can be continued smoothly in the variable x over the borderline of the domain $D(x_0)$. Thus, in virtue of (12), we obtain (for $t > 0$, $x \in \tilde{M}$)

$$\left(i \frac{\partial}{\partial t} + \Delta\right) Z(t, x, x_0) = -\frac{\partial}{\partial \vec{n}} (Z(t, x, x_0) \delta_{\partial D(x_0)}). \quad (19)$$

Remark. In (19) as well as everywhere in this section we use the following convention. The value of a density (which is in this case $Z(t, x, x_0)$) on the border $\partial D(x_0)$ is understood as the limit achieved from the interior of the domain $D(x_0)$.

Next we discuss the case $|\gamma| = 1$. Then γ designates a piecewise geodesic curve $x_0 \rightarrow C \rightarrow x$, with $C \in \mathcal{A} \cup \mathcal{B}$. Denote by γ' the geodesic segment $x_0 \rightarrow x$ (provided $x \in D(x_0)$). We have

$$\begin{aligned} \mathcal{K}_\gamma(t, x, x_0) &= \frac{1}{8\pi^2 i} \chi(x, C) \chi(C, x_0) \\ &\times \int_0^t \left(\left(\theta - \pi + i \log \left(\frac{(t-s)r_0}{sr} \right) \right)^{-1} - \left(\theta + \pi + i \log \left(\frac{(t-s)r_0}{sr} \right) \right)^{-1} \right) \\ &\times \frac{1}{(t-s)s} \exp \left(i \left(\frac{r^2}{4(t-s)} + \frac{r_0^2}{4s} \right) \right) ds \end{aligned} \quad (20)$$

where $r = \text{dist}(x, C)$, $r_0 = \text{dist}(C, x_0)$ and $\theta = \angle x_0, C, x$.

An application of the differential operator $(i\partial_t + \Delta)$ to the RHS of (20) in the sense of distributions results in several singular terms supported on one-dimensional submanifolds. First, due to the discontinuity of the characteristic function $\chi(x, C)$, the application of the Laplace operator leads to two terms supported on the boundary $\partial D(C)$ (see (12)). Second, as it follows from (11), the singularity of the integrand for the values $\theta = \pm\pi$ and $r_0/s = r/(t-s)$ produces terms supported on the submanifold determined by $\theta = \pm\pi$, and this set is nothing but a part of the boundary of the

domain $D(x_0)$, namely $\partial D(x_0; C)$. Notice that for $\theta = \pm\pi$ it holds $r + r_0 = \text{dist}(x, x_0)$ and $\partial/\partial \vec{n} = \pm r^{-1} \partial/\partial \theta$. Moreover, in the polar coordinates centered at C ,

$$\delta_{\partial D(x_0; C)} = \frac{1}{r} (\delta(\theta - \pi) + \delta(\theta + \pi)).$$

Thus the latter contribution takes the form

$$\frac{1}{4\pi i t r^2} \frac{\partial}{\partial \theta} \left(\exp\left(\frac{i}{4t} \text{dist}(x, x_0)^2\right) (\delta(\theta - \pi) - \delta(\theta + \pi)) \right) = \frac{\partial}{\partial \vec{n}} \left(\mathcal{K}_t^{\gamma'}(x, x_0) \delta_{\partial D(x_0, C)} \right)$$

where $\mathcal{K}_{\gamma'}(t, x, x_0) = Z(t, x, x_0)$. In summary, we obtain

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \Delta \right) \mathcal{K}_\gamma(t, x, x_0) &= - \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_\gamma(t, x, x_0) \right) \delta_{\partial D(G)} - \frac{\partial}{\partial \vec{n}} (\mathcal{K}_\gamma(t, x, x_0) \delta_{\partial D(C)}) \\ &\quad + \frac{\partial}{\partial \vec{n}} (\mathcal{K}_{\gamma'}(t, x, x_0) \delta_{\partial D(x_0, C)}). \end{aligned} \quad (21)$$

Finally let us consider the case $|\gamma| \geq 2$. Thus γ is a piecewise geodesic curve $x_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n \rightarrow x$, $n \geq 2$. Denote by γ' the truncated geodesic curve $x_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_{n-1} \rightarrow x$ (provided $x \in D(C_{n-1})$). One can express

$$\begin{aligned} \mathcal{K}_\gamma(t, x, x_0) &= \int_{\mathbb{R}^n} dt_{n-1} \dots dt_0 V \left(\begin{matrix} x, C_n, C_{n-1} \\ t - \tau, t_{n-1} \end{matrix} \right) Z(t - \tau, x, C_n) F_\gamma(t_0, \dots, t_{n-1}, x_0) \\ &= \frac{1}{2\pi} \chi(x, C_n) \int_{\mathbb{R}^{n-1}} dt_{n-2} \dots dt_0 \int_0^{t-\tau'} dt_{n-1} \\ &\quad \times \left(\frac{1}{\theta - \pi + i \log\left(\frac{(t-\tau'-t_{n-1})\varrho}{t_{n-1}r}\right)} - \frac{1}{\theta + \pi + i \log\left(\frac{(t-\tau'-t_{n-1})\varrho}{t_{n-1}r}\right)} \right) \\ &\quad \times \frac{1}{t - \tau' - t_{n-1}} \exp\left(i \frac{r^2}{4(t - \tau' - t_{n-1})}\right) F_\gamma(t_0, \dots, t_{n-1}, x_0) \end{aligned} \quad (22)$$

where

$$\tau = t_0 + \dots + t_{n-2} + t_{n-1}, \quad \tau' = t_0 + \dots + t_{n-2}, \quad r = \text{dist}(C_n, x), \quad \theta = \angle C_{n-1}, C_n, x,$$

and

$$F_\gamma(t_0, \dots, t_{n-1}, x_0) = \prod_{j=0}^{n-2} V \left(\begin{matrix} C_{j+2}, C_{j+1}, C_j \\ t_{j+1}, t_j \end{matrix} \right) \prod_{j=0}^{n-1} Z_{t_j}(C_{j+1}, C_j).$$

An application of the differential operator $(i\partial_t + \Delta)$ to the RHS of (22) in the sense of distributions again produces several singular terms. In consequence of the discontinuity of the characteristic function $\chi(x, C_n)$ a single and a double layer supported on the boundary $\partial D(C_n)$ occur (see (12)). The singularity of the integrand for the values $\theta = \pm\pi$ and $\varrho/t_{n-1} = r/(t - \tau' - t_{n-1})$ produces terms supported on the part of the boundary of the domain $D(C_{n-1})$, namely on $\partial D(C_{n-1}; C_n)$. This time one can apply identity (10). In order to treat the resulting terms the following equalities are useful.

Suppose that $\theta = \pm\pi$ and so $x \in \partial D(C_{n-1}; C_n)$. Set

$$r' = r + \varrho = \text{dist}(C_{n-1}, x), \quad \theta' = \angle C_{n-2}, C_{n-1}, x.$$

If $\varrho/t_{n-1} = r/(t - \tau' - t_{n-1})$ then

$$t_{n-1} = \frac{\varrho(t - \tau')}{r'} \quad \text{and} \quad \frac{t - \tau' - t_{n-1}}{r} = \frac{t - \tau'}{r'}.$$

Moreover,

$$\frac{\varrho}{r'} \exp\left(\frac{ir^2}{4(t - \tau)}\right) Z(t_{n-1}, C_n, C_{n-1}) = Z(t - \tau', x, C_{n-1})$$

and

$$V\left(\begin{array}{c} C_n, C_{n-1}, C_{n-2} \\ \varrho s_2/r', s_1 \end{array}\right) = V\left(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ s_2, s_1 \end{array}\right).$$

Observe also that

$$\begin{aligned} \frac{\partial}{\partial s} \left(\exp\left(\frac{ir^2}{4(t - \tau' - s)}\right) \exp\left(\frac{i\varrho^2}{4s}\right) \right) \Big|_{s=\varrho(t-\tau')/r'} &= 0, \\ \exp\left(\frac{ir^2}{4(t - \tau' - s)}\right) \exp\left(\frac{i\varrho^2}{4s}\right) \Big|_{s=\varrho(t-\tau')/r'} &= \exp\left(\frac{ir'^2}{4(t - \tau')}\right), \end{aligned}$$

and for $\theta = \pi$,

$$\frac{\partial}{\partial s} V\left(\begin{array}{c} C_n, C_{n-1}, C_{n-2} \\ s, t_{n-2} \end{array}\right) \Big|_{s=\varrho(t-\tau')/r'} = \frac{ir'}{\varrho(t - \tau')} \frac{\partial}{\partial \theta'} V\left(\begin{array}{c} x, C_{n-1}, C_{n-2} \\ t - \tau', t_{n-2} \end{array}\right).$$

A similar relation holds true also for $\theta = -\pi$.

After a bit tedious but quite straightforward manipulations one arrives at the final equality

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \Delta\right) \mathcal{K}_\gamma(t, x, x_0) &= - \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_\gamma(t, x, x_0)\right) \delta_{\partial D(C_n)} - \frac{\partial}{\partial \vec{n}} (\mathcal{K}_\gamma(t, x, x_0) \delta_{\partial D(C_n)}) \\ &\quad + \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_{\gamma'}(t, x, x_0)\right) \delta_{\partial D(C_{n-1}; C_n)} \\ &\quad + \frac{\partial}{\partial \vec{n}} (\mathcal{K}_{\gamma'}(t, x, x_0) \delta_{\partial D(C_{n-1}; C_n)}). \end{aligned} \tag{23}$$

Now we can show equality (14) when taking into account (19), (21) and (23). It is true that

$$\begin{aligned}
& \left(i \frac{\partial}{\partial t} + \Delta \right) \mathcal{K}(t, x, x_0) \\
&= \sum_{|\gamma| \geq 2} \left[- \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_\gamma(t, x, x_0) \right) \delta_{\partial D(C_n)} - \frac{\partial}{\partial \vec{n}} (\mathcal{K}_\gamma(t, x, x_0) \delta_{\partial D(C_n)}) \right. \\
&\quad \left. + \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_{\gamma'}(t, x, x_0) \right) \delta_{\partial D(C_{n-1}; C_n)} + \frac{\partial}{\partial \vec{n}} (\mathcal{K}_{\gamma'}(t, x, x_0) \delta_{\partial D(C_{n-1}; C_n)}) \right] \\
&+ \sum_{|\gamma|=1} \left[- \left(\frac{\partial}{\partial \vec{n}} \mathcal{K}_\gamma(t, x, x_0) \right) \delta_{\partial D(G)} - \frac{\partial}{\partial \vec{n}} (\mathcal{K}_\gamma(t, x, x_0) \delta_{\partial D(G)}) \right. \\
&\quad \left. + \frac{\partial}{\partial \vec{n}} (Z(t, x, x_0) \delta_{\partial D(x_0; C)}) \right] - \frac{\partial}{\partial \vec{n}} (Z(t, x, x_0) \delta_{\partial D(x_0)}) \\
&= 0
\end{aligned}$$

where we have used (16), (17) and (18).

4 The Aharonov-Bohm effect with two vortices: the propagator associated to H_Λ

Without loss of generality we can suppose that the vortices are located in the points $a = (0, 0)$ and $b = (\varrho, 0)$. Let (r_a, θ_a) be the polar coordinates centered at the point a and (r_b, θ_b) be the polar coordinates centered at the point b . To express the propagator for H_Λ it is convenient to pass to a unitarily equivalent formulation. Let us cut the plane along two half-lines,

$$L_a =] - \infty, 0[\times \{0\} \text{ and } L_b =] \varrho, +\infty[\times \{0\}.$$

The values $\theta_a = \pm\pi$ correspond to the two sides of the cut L_a , and similarly for θ_b and L_b . The unitarily equivalent Hamiltonian H'_Λ is formally equal to $-\Delta$ in $L^2(\mathbb{R}^2, d^2x)$ and is determined by the boundary conditions along the cut,

$$\begin{aligned}
\psi(r_a, \theta_a = \pi) &= e^{2\pi i \alpha} \psi(r_a, \theta_a = -\pi), \quad \partial_{r_a} \psi(r_a, \theta_a = \pi) = e^{2\pi i \alpha} \partial_{r_a} \psi(r_a, \theta_a = -\pi), \\
\psi(r_b, \theta_b = \pi) &= e^{2\pi i \beta} \psi(r_b, \theta_b = -\pi), \quad \partial_{r_b} \psi(r_b, \theta_b = \pi) = e^{2\pi i \beta} \partial_{r_b} \psi(r_b, \theta_b = -\pi).
\end{aligned}$$

In addition, one should impose a boundary condition at the vortices, namely $\psi(a) = \psi(b) = 0$.

Let us denote $D = \mathbb{R}^2 \setminus (L_a \cup L_b)$. Then one can embed $D \subset \tilde{M}$ as a fundamental domain. We wish to find a formula for the propagator $\mathcal{K}'^\Lambda(t, x, x_0)$ associated to the Hamiltonian H'_Λ . It can be simply obtained as the restriction to D of the propagator $\mathcal{K}^\Lambda(t, x, x_0)$ associated to the Hamiltonian H_Λ . On the other hand, to construct $\mathcal{K}^\Lambda(t, x, x_0)$ one can apply formula (3) and the knowledge of the free propagator on \tilde{M} , see (5), (6). Thus we get

$$\mathcal{K}^\Lambda(t, x, x_0) = \sum_{g \in \Gamma} \sum_{\gamma} \Lambda(g^{-1}) \mathcal{K}_\gamma(t, g \cdot x, x_0). \quad (24)$$

Fix $t > 0$ and $x_0, x \in D$. One can classify piecewise geodesic paths in \tilde{M} ,

$$\gamma : x_0 \rightarrow C_1 \rightarrow \dots \rightarrow C_n \rightarrow g \cdot x, \quad (25)$$

with $C_j \in \mathcal{A} \cup \mathcal{B}$ and $g \in \Gamma$, according to their projections to M . Let $\bar{\gamma}$ be a finite alternating sequence of points a and b , i.e., $\bar{\gamma} = (c_1, \dots, c_n)$, $c_j \in \{a, b\}$ and $c_j \neq c_{j+1}$. The empty sequence $\bar{\gamma} = ()$ is admissible. Relate to $\bar{\gamma}$ a piecewise geodesic path in M , namely $x_0 \rightarrow c_1 \rightarrow \dots \rightarrow c_n \rightarrow x$. Suppose that this path is covered by a path γ in \tilde{M} , as given in (25). Then $C_j \in \mathcal{A}$ iff $c_j = a$ and $C_j \in \mathcal{B}$ iff $c_j = b$. Denote the angles $\angle x_0, c_1, c_2 = \theta_0$ and $\angle c_{n-1}, c_n, x = \theta$. Then the angles in the path γ in (25) take the values $\angle x_0, C_1, C_2 = \theta_0 + 2\pi k_1$, $\angle C_{n-1}, C_n, g \cdot x = \theta + 2\pi k_n$ and $\angle C_j, C_{j+1}, C_{j+2} = 2\pi k_{j+1}$ for $1 \leq j \leq n-2$ (if $n \geq 3$), where k_1, \dots, k_n are integers. Any values $k_1, \dots, k_n \in \mathbb{Z}$ are possible. In that case the representation Λ applied to the group element g occurring in (25) takes the value

$$\Lambda(g) = \exp(2\pi i(k_1\sigma_1 + \dots + k_n\sigma_n))$$

where $\sigma_j \in \{\alpha, \beta\}$ and $\sigma_j = \alpha$ if $c_j = a$, and $\sigma_j = \beta$ if $c_j = b$.

Using the equalities

$$\begin{aligned} & \sum_{k \in \mathbb{Z}} \exp(2\pi i \alpha k) \left(\frac{1}{\theta + 2k\pi - \pi + i s} - \frac{1}{\theta + 2\pi k + \pi + i s} \right) \\ &= -\sin(\pi \alpha) \int_{-\infty}^{+\infty} \frac{\exp((\theta + i s)\tau)}{\sin(\pi(\alpha + i \tau))} d\tau \end{aligned}$$

and

$$\int_{-\infty}^{+\infty} \frac{\exp((\theta + i s)\tau)}{\sin(\pi(\alpha + i \tau))} d\tau = 2 \frac{\exp(-\alpha(s - i\theta))}{1 + \exp(-s + i\theta)},$$

that are valid for $0 < \alpha < 1$, $|\theta| < \pi$, one can carry out a partial summation in (24) over the integers k_1, \dots, k_n . This way the double sum in (24) reduces to a sum over finite alternating sequences $\bar{\gamma}$.

Let us conclude our contribution by giving the resulting formula for $\mathcal{K}'^\Lambda(t, x, x_0)$. We set

$$\zeta_a = 1 \text{ or } \zeta_a = e^{2\pi i \alpha} \text{ or } \zeta_a = e^{-2\pi i \alpha}$$

depending on whether the segment $\overline{x_0 x}$ does not intersect L_a , or $\overline{x_0 x}$ intersects L_a and x_0 lies in the lower half-plane, or $\overline{x_0 x}$ intersects L_a and x_0 lies in the upper half-plane. Analogously,

$$\zeta_b = 1 \text{ or } \zeta_b = e^{2\pi i \beta} \text{ or } \zeta_b = e^{-2\pi i \beta}$$

depending on whether the segment $\overline{x_0 x}$ does not intersect L_b , or $\overline{x_0 x}$ intersects L_b and x_0 lies in the upper half-plane, or $\overline{x_0 x}$ intersects L_b and x_0 lies in the lower half-plane. Furthermore, let us set

$$\zeta_a = e^{i\alpha\eta_a}, \quad \zeta_b = e^{i\beta\eta_b}, \quad \text{where } \eta_a, \eta_b \in \{0, 2\pi, -2\pi\}.$$

Then one has

$$\begin{aligned}
& \mathcal{K}'^\Lambda(t, x, x_0) \\
&= \zeta_a \zeta_b \frac{1}{4\pi i t} \exp\left(i \frac{|x - x_0|^2}{4t}\right) \\
&\quad - \zeta_a \frac{\sin(\pi\alpha)}{4\pi^2 i} \int_0^\infty \frac{dt_1}{t_1} \int_0^\infty \frac{dt_0}{t_0} \delta(t_1 + t_0 - t) \\
&\quad \quad \times \exp\left(i \left(\frac{r_a^2}{4t_1} + \frac{r_{0a}^2}{4t_0}\right)\right) \frac{\exp[-\alpha(s_a - i(\theta_a - \theta_{0a} - \eta_a))]}{1 + \exp(-s_a + i\theta_a - i\theta_{0a})} \\
&\quad - \zeta_b \frac{\sin(\pi\beta)}{4\pi^2 i} \int_0^\infty \frac{dt_1}{t_1} \int_0^\infty \frac{dt_0}{t_0} \delta(t_1 + t_0 - t) \\
&\quad \quad \times \exp\left(i \left(\frac{r_b^2}{4t_1} + \frac{r_{0b}^2}{4t_0}\right)\right) \frac{\exp[-\beta(s_b - i(\theta_b - \theta_{0b} - \eta_b))]}{1 + \exp(-s_b + i\theta_b - i\theta_{0b})} \\
&\quad + \frac{1}{4\pi i} \sum_{\bar{\gamma}, n \geq 2} (-1)^n \int_0^\infty \frac{dt_n}{t_n} \dots \int_0^\infty \frac{dt_0}{t_0} \delta(t_n + \dots + t_0 - t) \\
&\quad \quad \times \exp\left(\frac{i}{4} \left(\frac{r^2}{t_n} + \frac{\varrho^2}{t_{n-1}} + \dots + \frac{\varrho^2}{t_1} + \frac{r_0^2}{t_0}\right)\right) S_{\bar{\gamma}}(s, \theta, \theta_0),
\end{aligned}$$

where

$$\begin{aligned}
S_{\bar{\gamma}}(s, \theta, \theta_0) &= \frac{\sin(\pi\sigma_n)}{\pi} \frac{\exp[-\sigma_n(s_n - i\theta)]}{1 + \exp(-s_n + i\theta)} \frac{\sin \pi\sigma_{n-1}}{\pi} \frac{\exp(-\sigma_{n-1}s_{n-1})}{1 + \exp(-s_n)} \\
&\quad \times \dots \times \frac{\sin(\pi\sigma_1)}{\pi} \frac{\exp[-\sigma_1(s_1 - i\theta_0)]}{1 + \exp(-s_1 + i\theta_0)},
\end{aligned}$$

and

$$s_a = \log\left(\frac{t_1 r_{0a}}{t_0 r_a}\right), \quad s_b = \log\left(\frac{t_1 r_{0b}}{t_0 r_b}\right), \quad s_j = \log\left(\frac{t_j r_{j-1}}{t_{j-1} r_j}\right) \quad \text{for } 1 \leq j \leq n.$$

In addition, (r, θ) are the polar coordinates of the point x with respect to the center c_n , (r_0, θ_0) are the polar coordinates of the point x_0 with respect to the center c_1 . The sum $\sum_{\bar{\gamma}, n \geq 2}$ runs over all finite alternating sequences of length at least two, $\bar{\gamma} = (c_1, \dots, c_n)$, such that for all j , $c_j \in \{a, b\}$, $c_j \neq c_{j+1}$, and $\sigma_j = \alpha$ (resp. β) depending on whether $c_j = a$ (resp. b).

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